A THEOREM IN DIFFERENCE EQUATIONS ON THE ALTERNATION OF NODES OF LINEARLY INDEPENDENT SOLUTIONS.

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Consider the difference equation

$$(1) \qquad L(i)u(i+1) + M(i)u(i) + N(i)u(i-1) = 0 \quad (i = 1, 2, \cdots, n).$$

Plot a solution $u(i)$ of this equation on an X-axis, making $u(i)$ correspond to a point x_i, where $x_i < x_{i+1}$, and joining successive points by straight line segments; a point x at which this broken line meets the axis will be called a *node* of the solution. Concerning the nodes of linearly independent solutions, the following analogon of a classic theorem of Sturm for differential equations may be stated:

Theorem. *For any linear homogeneous difference equation of the form* (1), *where* $L(i)N(i) > 0$ *for every value of* i $(i = 1, 2, \cdots, n)$, *the nodes of any two linearly independent solutions separate each other.**

The proof of this theorem can be effected as follows. Let $u_1(i)$ and $u_2(i)$ be any two linearly independent solutions. Then

1°. No node of $u_1(i)$ coincides with a node of $u_2(i)$. For if x were a node of both $u_1(i)$ and $u_2(i)$ it would be a node of every solution; but supposing $x_k \leqq x \leqq x_{k+1}$, it is obvious that the solution determined by the conditions $u(k) = 1$, $u(k+1) = 1$ has no node at x.

Next, using the notation

$$(2) \qquad W(k) \equiv u_1(k+1)u_2(k) - u_1(k)u_2(k+1) \quad (k = 0, 1, \cdots, n),$$

it is known from the general theory of difference equations that $W(k)$ is distinct from zero; and one may prove

2°. For any values of k and l such that $W(k)$ and $W(k+l)$ are defined,

* The proof of this theorem was given for an exercise by Professor Bôcher in a course in differential equations at Harvard; and proofs more or less similar to the one here given were worked out simultaneously with and independently of this one by Messrs. Allen, Brand, Fort and Graustein. A theorem very similar to this one is given by Porter in these Annals, 2 Series, vol. 3, 1901–1902, on p. 65. He makes the assumption that neither of the solutions considered has a node at x_0. And his proof, which is very different in character from the one here given, depending upon the introduction of an auxiliary parameter, seems to be lacking in completeness; it does not seem obvious, for example, that the v-points (i. e., nodes) all vary in one direction as the parameter z increases.

$$\frac{W(k)}{W(k+l)} > 0.$$

It is no restriction to assume $l \geqq 0$; and for $l = 0$ the result is immediate. From the identities

$$L(k+1)u_1(k+2) + M(k+1)u_1(k+1) + N(k+1)u_1(k) = 0,$$
$$L(k+1)u_2(k+2) + M(k+1)u_2(k+1) + N(k+1)u_2(k) = 0,$$

one easily obtains

$$L(k+1)W(k+1) - N(k+1)W(k) = 0.$$

Hence, since $W(k+1) \neq 0$ and $L(k+1)N(k+1) > 0$,

$$\frac{W(k)}{W(k+1)} = \frac{L(k+1)}{N(k+1)} > 0;$$

and therefore

$$\frac{W(k)}{W(k+l)} = \prod_{j=1}^{l} \frac{L(k+j)}{N(k+j)} > 0.$$

The theorem is without content unless at least one of the solutions has two or more nodes on the interval $x_0 \leqq x \leqq x_{n+1}$. Suppose then that $u_1(i)$ has two successive nodes at x' and x'':

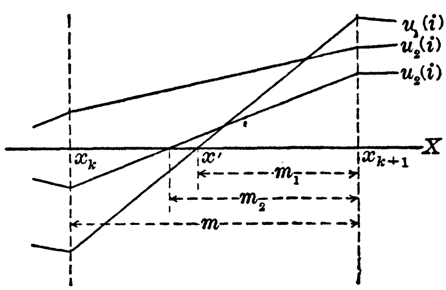
$$x_k \leqq x' < x_{k+1}, \quad x_{k+l} < x'' \leqq x_{k+l+1}, \quad l \geqq 1.$$

One may prove next

3°. If $u_2(i)$ has no node on the interval $x' < x < x''$ then

$$(3) \qquad a)\ \frac{W(k)}{u_1(k+1)u_2(k+1)} > 0; \quad b)\ \frac{W(k+l)}{u_1(k+l)u_2(k+l)} < 0.$$

Consider the interval $x_k \leqq x \leqq x_{k+1}$; the possibilities for $u_2(i)$ are indicated in the figure; it is to be observed that if the signs of $u_1(i)$ or of $u_2(i)$



or of both $u_1(i)$ and $u_2(i)$ are changed the following argument is not altered. Now $u_1(k+1)$ and $u_2(k+1)$ are not zero and hence from (2)

$$\frac{W(k)}{u_1(k+1)u_2(k+1)} = \frac{u_2(k)}{u_2(k+1)} - \frac{u_1(k)}{u_1(k+1)}.$$

If $u_1(k) = 0$, then, using 1°,

$$\frac{u_2(k)}{u_2(k+1)} > 0;$$

hence (3*a*). If $u_1(k) \neq 0$, then either

$$\frac{u_2(k)}{u_2(k+1)} \geqq 0, \quad \frac{-u_1(k)}{u_1(k+1)} > 0,$$

and hence (3*a*); or, referring to the figure, and using 1°,

$$\frac{u_2(k)}{u_2(k+1)} - \frac{u_1(k)}{u_1(k+1)} = \left(-\frac{m-m_2}{m_2}\right) - \left(-\frac{m-m_1}{m_1}\right) = -m\frac{m_1-m_2}{m_1m_2} > 0.$$

Therefore (3*a*) holds in all cases. (3*b*) can be established in a similar manner.

Now suppose that $u_2(i)$ has no node on the interval $x' < x < x''$; then dividing (3*a*) by (3*b*),

$$\frac{W(k)}{W(k+l)} \cdot \frac{u_1(k+l)}{u_1(k+1)} \cdot \frac{u_2(k+l)}{u_2(k+1)} < 0.$$

Hence by 2° and the hypothesis that $u_1(i)$ has no node on the interval $x' < x < x''$, it follows that

$$\frac{u_2(k+l)}{u_2(k+1)} < 0.$$

If $l = 1$ this leads at once to a contradiction; if $l > 1$ this implies that $u_2(i)$ has a node on $x_{k+1} \leqq x \leqq x_{k+l}$, and one is again led to a contradiction. Hence between every pair of nodes of one solution there is a node of the other; that is, the nodes separate each other.

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